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Painlevé solution of an integral equation

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Abstract
It is demonstrated that a certain integral equation can be solved using the Painlevé equation of third kind. Inversely, a special solution of this Painlevé equation can be expressed as the ratio of two infinite series of spheroidal functions with known coefficients.

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1. Introduction

We consider an integral equation in a finite interval for an unknown function $g(t)$:

$$\int_{-1}^{1} K(|x-t|)g(t)\,\mathrm{d}t = f(x), \tag{1}$$

where the kernel $K(w)$ has the form

$$K(w) = w^{\nu} K_{\nu}(\theta w). \tag{2}$$

Here, $K_{\nu}(w)$ is the modified Bessel function of the third kind, which obeys the equation

$$K_{\nu}''(w) + \frac{1}{w}K_{\nu}'(w) - \left(1 + \frac{\nu^2}{w^2}\right)K_{\nu}(w) = 0 \tag{3}$$

and exponentially decreases at a large argument

$$K_{\nu}(w) \underset{w\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{2w}}\,\mathrm{e}^{-w}. \tag{4}$$

At small w

$$K_{\nu}(w) \underset{w\to 0}{\longrightarrow} 2^{\nu-1}\Gamma(\nu)w^{-\nu} + 2^{-\nu-1}\Gamma(-\nu)w^{\nu} + \mathrm{O}(w^{2-|\nu|}). \tag{5}$$

We restrict the discussion to the case $|\nu| < 1/2$.

Equation (1) with kernel (2) appeared in some physical problems. For example, it describes forced convection heat transfer and contact problems in elasticity (see [1, 2] and references therein). The particular case of (1) with $\nu = 0$ and imaginary θ (i.e. with $H_0^{(1)}(\theta w)$ kernel instead of (2)) gives the solution of the wave-scattering problem with the Dirichlet boundary conditions at the strip $[-1, 1]$ (see, e.g., [3]). Usually, such problems are defined by partial differential equations. Their reduction to an integral equation (1) simplifies the problem but still requires the numerical inversion of a large matrix or the development of perturbative expansions at small or large values of θ (see, e.g., [3]).

In [1], the solution of this equation has been represented as a series of the spheroidal functions, which generalizes the expansion in terms of Mathieu functions for the case $\nu = 0$ obtained in [4]. Equation (1) belongs to the class of equations whose particular solutions can be obtained by Latta's method [5] but the resulting equations contain unknown constants. For $\nu = 0$, these constants have been explicitly calculated by Myers [6] through a solution of the Painlevé equation of third kind (a concise summary of Myers' result is given in appendix B of [7]).

The purpose of this paper is to show that methods used for $\nu = 0$ can be generalized for nonzero ν ($|\nu| < 1/2$) and special solutions of (1) can be obtained from ordinary differential equations (ODE) using the Painlevé III equation more general than the one discussed in [6]. As a by-product, we obtain an explicit formula for a solution of the Painlevé III equation as a ratio of two infinite series of spheroidal functions with the known coefficients. The obtained formulae require numerical solutions of a few ODEs with known initial conditions and may not be advantageous to the direct numerical approach to (1) but they are exact and give unexpected relations between different problems.

The plan of this paper is the following. In section 2, we use Latta's method to obtain a system of ODE in t variable for two special solutions of (1) corresponding to the right-hand side of (1) equal to $\cosh\theta x$ and $\sinh\theta x$. These equations contain certain constants, which are determined in section 3 by deriving the second system of ODEs with respect to variable θ. The condition of the compatibility of these two systems of equations leads to the Painlevé III equation for the constants. The limiting behaviour of necessary quantities for small and large θ are derived in appendix C. Although the discussed methods work only for the above-mentioned particular choices of the right-hand side of (1), in section 4, it is demonstrated that the knowledge of these special solutions permits to investigate more general problems. In section 5, we show that kernel (2) is a positive-definite function, which leads to the boundedness of the discussed Painlevé III solution at positive arguments. The main steps to implement the obtained solution are briefly reviewed in section 6. For completeness, in appendix A, the series expansion of solutions of integral equation (1) is present, and in appendix B, properties of spheroidal functions are briefly described.

2. First system of equations

A few exact approaches for solving integral equation (1) for different kernels $K(w)$ have been discussed in the literature [1, 4–6, 8–11]. The method of Latta [5] can be applied for integral equations of the form (1) when its kernel $K(w)$ obeys a differential equation whose coefficients depends linearly on w. The kernel $K(w)$ determined in (2) obeys the equation

$$wK''(w) + (1-2\nu)K'(w) - w\theta^2 K(w) = 0 \tag{6}$$

so the application of Latta's method is straightforward.

Let us define the integral operator corresponding to (1)

$$(\Gamma g)(x) \equiv \int_{-1}^{1} K(|x-t|)g(t)\,\mathrm{d}t \tag{7}$$

and its two particular solutions $g_c(t)$ and $g_s(t)$, with the right-hand side, equal to $\cosh(\theta x)$ and $\sinh(\theta x)$:

$$(\Gamma g_c)(x) = \cosh(\theta x), \quad (\Gamma g_s)(x) = \sinh(\theta x). \tag{8}$$

It is plain that parities of $g_{c,s}(t)$ are fixed: $g_c(-t) = g_c(t)$ and $g_s(-t) = -g_s(t)$.

An important ingredient of Latta's method is the uniqueness of the solution: when $f(x) = 0$ the only solution of (1) is $g(t) = 0$. For kernels like (2), it is known that if $f(x)$ belongs to $L_2(-1, 1)$, then the solution is unique (see section 5). An other way to prove it is to use the expansion into a complete set of functions as in appendix A.

Changing in (6) the variable w to $x - t$, multiplying the resulting expression by $g(t)$ and integrating over $t \in [-1, 1]$ gives the following identity (the first Latta equation):

$$(\Gamma tg)''(x) - \theta^2(\Gamma tg)(x) = x[(\Gamma g)''(x) - \theta^2(\Gamma g)(x)] + (1 - 2\nu)(\Gamma g)'(x) = 0, \tag{9}$$

which permits to calculate (Γtg) when the function $f(x) \equiv (\Gamma g)(x)$ is known and obeys the equation $f'' - \theta^2 f = 0$.

The second Latta equation is obtained by remarking that the kernel in (1) depends only on the difference of arguments. Therefore, it is easy to check that for an arbitrary function $y(t)$, such that $y(\pm 1) = 0$,

$$(\Gamma y)'(x) = (\Gamma y')(x). \tag{10}$$

To apply this equation, it is necessary first to determine the behaviour of function $g(t)$ near ends of the strip, $t = \pm 1$. Collecting the most singular terms as it is done in appendix C or using the representation (A.5) gives

$$g_{c,s} \underset{t\to 1}{\longrightarrow} \frac{k_{c,s}}{(1-t)^{\nu+1/2}}. \tag{11}$$

Therefore, when $\nu < 1/2$, functions

$$y_{c,s}(t) = (1 - t^2)g_{c,s}(t) \tag{12}$$

are zero at the both ends of the strip, $y_{c,s}(\pm 1) = 0$ and (10) is valid for them.

Equations (9) and (10) permit to derive ODEs for special solutions (8) as follows. By definition

$$(\Gamma g_c)(x) = \cosh\theta x. \tag{13}$$

From (9), it follows that $y = (\Gamma tg_c)$ obeys the equation

$$y'' - \theta^2 y = (1 - 2\nu)\theta \sinh\theta x \tag{14}$$

whose odd solution is

$$y \equiv (\Gamma tg_c)(x) = \frac{1-2\nu}{2} x\cosh\theta x + A\sinh\theta x, \tag{15}$$

where A is a constant independent of x.

In exactly the same manner, the knowledge of Γtg_c permits to calculate $\Gamma t^2 g_c$:

$$(\Gamma t^2 g_c)(x) = \frac{(1-2\nu)(3-2\nu)}{8}x^2\cosh\theta x + x\sinh\theta x\left(\frac{1-2\nu}{2}A - \frac{1-4\nu^2}{8\theta}\right) + C\cosh\theta x, \tag{16}$$

where C is another constant.

Similarly, with certain constants B and D,

$$(\Gamma g_s)(x) = \sinh\theta x, \qquad (\Gamma tg_s)(x) = \frac{1-2\nu}{2}x\sinh\theta x + B\cosh\theta x \tag{17}$$

and

$$(\Gamma t^2 g_s)(x) = \frac{(1-2\nu)(3-2\nu)}{8}x^2\sinh\theta x + x\cosh\theta x\left(\frac{1-2\nu}{2}B - \frac{1-4\nu^2}{8\theta}\right) + D\sinh\theta x. \tag{18}$$

Using (10), one obtains

$$(\Gamma\,[(1-t^2)g_c]')(x) = (\Gamma\,[(1-t^2)g_c])'(x). \tag{19}$$

From the above relations, it follows that

$$(\Gamma\,[(1-t^2)g_c]')(x) = -\frac{(1-2\nu)(3-2\nu)}{8}[\theta x^2\sinh\theta x + 2x\cosh\theta x] - \left(\frac{1-2\nu}{2}A - \frac{1-4\nu^2}{8\theta}\right)[\theta x\cosh\theta x + \sinh\theta x] + (1-C)\theta\sinh\theta x. \tag{20}$$

Expressing the right-hand side through $(\Gamma\,t^2 g_s)$, $(\Gamma t g_c)$ and (Γg_s), one obtains

$$(\Gamma\,[(1-t^2)g_c]')(x) = -\theta(\Gamma\,t^2 g_s)(x) + \left(\theta(B-A) - \frac{3-2\nu}{2}\right)(\Gamma t g_c)(x) + \rho_1\,(\Gamma g_s)(x), \tag{21}$$

where ρ_1 is a certain combination of constants A, B and C.

As the solution of (1) is unique, the last equation implies that g_c and g_s have to obey the following ODE:

$$(1-t^2)g_c'(t) = \left(\nu + \tfrac{1}{2} - \rho\right)t g_c(t) + (\rho_1 - \theta t^2)g_s(t), \tag{22}$$

with $\rho = (A-B)\theta + 2$.

Repeating these calculations for g_s gives another equation

$$(1-t^2)g_s'(t) = \left(\nu + \tfrac{1}{2} + \rho\right)t g_s(t) + (\rho_2 - \theta t^2)g_c(t). \tag{23}$$

Imposing the condition that near the end $t = 1$, the functions $g_c(t)$ and $g_s(t)$ have the prescribed singularities (cf. (11))

$$g_c(t) \underset{t\to 1}{\longrightarrow} \frac{k_c}{(1-t)^{\nu+1/2}}, \qquad g_s(t) \underset{t\to 1}{\longrightarrow} \frac{k_c\,\eta}{(1-t)^{\nu+1/2}}, \tag{24}$$

where η is determined from the limit

$$\eta \equiv \frac{k_s}{k_c} = \lim_{t\to 1}\frac{g_s(t)}{g_c(t)}, \tag{25}$$

one fixes constants ρ_1 and ρ_2.

Finally, we conclude that for $|\nu| < 1/2$, the functions $g_c(t)$ and $g_s(t)$ obey the system of ODEs

$$\frac{\partial}{\partial t}\begin{pmatrix} g_c \\ g_s \end{pmatrix} = M\begin{pmatrix} g_c \\ g_s \end{pmatrix} \tag{26}$$

with the following 2×2 matrix M:

$$M = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} + \frac{1}{1-t^2}\begin{pmatrix} t(\nu+1/2-\rho) & \frac{1}{\eta}(\nu+1/2+\rho) \\ \eta(\nu+1/2-\rho) & t(\nu+1/2+\rho) \end{pmatrix}. \tag{27}$$

When $\nu = 0$, these equations coincide with the ones in [5].

For the further use, we rewrite the system (26) in the form

$$\begin{aligned} \frac{\partial}{\partial t}\left[g_c - \frac{t}{\eta}g_s\right] &= -\frac{\theta t}{\eta}\,g_c + \left(\frac{\nu - \frac{1}{2} + \rho}{\eta} + \theta\right) g_s, \\ \frac{\partial}{\partial t}[g_s - t\eta\,g_c] &= -\theta t\eta\,g_s + \left(\eta\left(\nu - \frac{1}{2} - \rho\right) + \theta\right) g_c. \end{aligned} \tag{28}$$

The terms in the square brackets are zero when $t = \pm 1$ and this system can be obtained directly by using (10).

3. Second system of equations

To find constants ρ and η, we use scaling arguments similar but different to the ones discussed in [6].

The functions g_c and g_s in (8) depend on t and θ, $g_{c,s} = g_{c,s}(t,\theta)$. We are interesting in finding equations governing the evolution of these function with changing θ.

Consider instead of system (8) a more general system of equations

$$\Gamma G_c \equiv \int_{L_1}^{L_2} |x-t|^{\nu} K_{\nu}(k|x-t|) G_c(t)\,\mathrm{d}t = \cosh\left[k\left(x-\frac{1}{2}(L_1+L_2)\right)\right],$$
$$\Gamma G_s \equiv \int_{L_1}^{L_2} |x-t|^{\nu} K_{\nu}(k|x-t|) G_s(t)\,\mathrm{d}t = \sinh\left[k\left(x-\frac{1}{2}(L_1+L_2)\right)\right]. \quad (29)$$

Changing the variable

$$t = \tfrac{1}{2}(L_1+L_2) + \tfrac{1}{2}(L_2-L_1)y \quad (30)$$

together with the similar change of x, it is straightforward to check that the functions $G_c(y)$ and $G_s(y)$ are expressed through $g_c(t)$ and $g_s(t)$ defined in (8) as follows:

$$G_{c,s}(y) = \left(\frac{2}{L_2-L_1}\right)^{\nu+1} g_{c,s}(z,\theta), \quad (31)$$

where

$$z = \frac{2}{L_2-L_1}y - \frac{L_2+L_1}{L_2-L_1}, \qquad \theta = \frac{1}{2}k(L_2-L_1). \quad (32)$$

Consider two functions

$$\Psi_1(y) = G_s(y) + \eta G_c(y), \qquad \Psi_2(y) = G_s(y) - \eta G_c(y), \quad (33)$$

where η is the same as in (25). By construction, $\Psi_1(L_1) = 0$ and $\Psi_2(L_2) = 0$.

From (29), it follows that

$$\Gamma\Psi_{1,2} = \Gamma G_s \pm \eta \Gamma G_c = \sinh\left(k(x-\tfrac{1}{2}(L_1+L_2)\right) \pm \eta\cosh\left(k(x-\tfrac{1}{2}(L_1+L_2)\right). \quad (34)$$

As $\Psi_i(L_i) = 0$, the differentiation over L_i of these equations gives

$$\frac{\partial}{\partial L_i}(\Gamma\Psi_i) = \Gamma\left(\frac{\partial}{\partial L_i}\Psi_i\right), \qquad i{=}1,2. \quad (35)$$

Using the uniqueness of the solutions, after simple algebra, we obtain

$$\frac{\partial G_s}{\partial L_1} + \eta\frac{\partial G_c}{\partial L_1} = -\frac{\theta}{L_2-L_1}(G_c+\eta G_s),$$
$$\frac{\partial G_s}{\partial L_2} - \eta\frac{\partial G_c}{\partial L_2} = -\frac{\theta}{L_2-L_1}(G_c-\eta G_s). \quad (36)$$

Performing the calculations, one finds

$$\frac{\partial G_{c,s}}{\partial L_i} = \frac{(-1)^{i+1}}{L_2-L_1}\left(\frac{2}{L_2-L_1}\right)^{\nu+1}\left((1+\nu)g_{c,s} + (z+(-1)^i)\frac{\partial g_{c,s}}{\partial z} - \theta\frac{\partial g_{c,s}}{\partial\theta}\right), \quad (37)$$

where $i = 1$ for G_c and $i = 2$ for G_s.

Combining all terms together, one obtains that (36) take the form

$$\theta\eta\frac{\partial g_c}{\partial\theta} = -\left(\frac{\partial g_s}{\partial z} - \eta z\frac{\partial g_c}{\partial z}\right) + (\theta+\eta(1+\nu))g_c,$$
$$\theta\frac{\partial g_s}{\partial\theta} = -\left(z\frac{\partial g_s}{\partial z} - \eta\frac{\partial g_c}{\partial z}\right) + (\theta\eta+(1+\nu))g_s. \quad (38)$$

Using (28), we conclude that the functions $g_{c,s}(t,\theta)$ obey the following system of equations:

$$\frac{\partial}{\partial\theta}\begin{pmatrix} g_c \\ g_s \end{pmatrix} = N\begin{pmatrix} g_c \\ g_s \end{pmatrix}, \tag{39}$$

where the matrix N has the form

$$N = \begin{pmatrix} \frac{1/2+\rho}{\theta} & t \\ t & \frac{1/2-\rho}{\theta} \end{pmatrix}. \tag{40}$$

The condition of compatibility of systems of equations (27) and (39) gives the equations of zero curvature for the matrices N and M:

$$\frac{\partial}{\partial t}N - \frac{\partial}{\partial\theta}M = MN - NM. \tag{41}$$

Direct calculations prove that this equation will be valid provided $\eta = \eta(\theta)$ and $\rho = \rho(\theta)$ fulfil the equations

$$\rho = \frac{\theta}{2\eta}(1 - \eta' - \eta^2) \tag{42}$$

and

$$\rho' = \frac{\nu + 1/2 + \rho}{\eta} - (\nu + 1/2 - \rho)\eta. \tag{43}$$

Substituting here the previous equation, one finds that η has to obey the equation

$$\frac{d^2\eta}{d\theta^2} = \eta^{-1}\left(\frac{d\eta}{d\theta}\right)^2 - \theta^{-1}\frac{d\eta}{d\theta} - \frac{2\nu(1-\eta^2)}{\theta} + \eta^3 - \frac{1}{\eta}. \tag{44}$$

This equation is a particular one-parameter family of the Painlevé III equation exactly the same which has been studied in [13].

From the results of appendices A and B, it follows that the functions $g_c(\cos\gamma)$ and $g_s(\cos\gamma)$ can be written as the following series of the spheroidal functions:

$$g_c(\cos\gamma) = \frac{1}{(\sin\gamma)^{2\nu+1}}\sum_{m=0}^{\infty}\mu_{2m}\tilde{X}_{2m}(0)Y_{2m}(\gamma), \tag{45}$$

$$g_s(\cos\gamma) = \frac{1}{(\sin\gamma)^{2\nu+1}}\sum_{m=0}^{\infty}\mu_{2m+1}\tilde{X}_{2m+1}(0)Y_{2m+1}(\gamma), \tag{46}$$

where $Y_m(\cos\gamma)$ are defined in (B.1), $X_m(\xi)$ in (B.18) and μ_n in (A.24).

The function η in (25), which is a solution of the Painlevé III equation, is given as the ratio of the above series:

$$\eta(\theta) = \frac{\sum_{m=0}^{\infty}\mu_{2m+1}(\theta)\tilde{X}_{2m+1}(0)Y_{2m+1}(0)}{\sum_{m=0}^{\infty}\mu_{2m}(\theta)\tilde{X}_{2m}(0)Y_{2m}(0)}. \tag{47}$$

In order to use the equations derived in the previous sections, it is necessary to know the behaviour of the functions $g_c(t)$ and $g_s(t)$ defined in (8) in the limit of small θ and (or) large θ. This is done in appendix C.

4. Embedding formulae

In the previous sections, the integral equation (1) has been transformed into two systems of ODEs (26), (27) and (39), (40) but only for very special right-hand sides of this equation, $f(x) = \cosh(\theta x)$ and $f(x) = \sinh(\theta x)$ (cf. (8)). Nevertheless, the knowledge of these two special solutions permits to find solutions of (1) for more general cases [7], [10].

Let us consider the equation of the form

$$(\Gamma g)(x) = \mathrm{e}^{-\theta z x} \tag{48}$$

with the same kernel as in (2). To find its solution, i.e. $g(t)$, it is convenient to define symmetric and antisymmetric combinations of $g_{c,s}$:

$$g_{\pm}(t) = g_c(t) \pm g_s(t). \tag{49}$$

We look for a solution of (48) in the form

$$\Psi(t) = g(t) + a_+ g_+(t) + a_- g_-(t), \tag{50}$$

where a_+ and a_- are the two constants independent of t, which have to be determined from two conditions:

$$\lim_{t\to\pm 1} \Psi(t) = 0. \tag{51}$$

The explicit form of these constants will be presented later (see (63)).

Assuming that (51) is fulfilled, one concludes as in section 2 that

$$(\Gamma\Psi(x))' = (\Gamma\Psi')(x). \tag{52}$$

The right-hand side of this equation is calculated from the definition of Ψ and

$$(\Gamma\Psi(x))' = -\theta z\,\mathrm{e}^{-\theta z x} + \theta a_+\,\mathrm{e}^{\theta x} - \theta a_-\,\mathrm{e}^{-\theta x} = (\Gamma[-z\theta g + a_+\theta g_+ - a_-\theta g_-])(x). \tag{53}$$

Due to the uniqueness of the solution, $\Psi(t)$ obeys the equation

$$\Psi'(t) = -z\theta(\Psi(t) - a_+g_+(t) - a_-g_-(t)) + a_+\theta g_+(t) + a_-\theta g_-(t), \tag{54}$$

which is equivalent to

$$\theta^{-1}\Psi'(t) + z\Psi(t) = (z+1)a_+g_+(t) + (z-1)a_-g_-(t) \tag{55}$$

whose solution is straightforward if constants $a_\pm$ are known.

To find them, one can proceed as follows [10]. Let us calculate the following quantities:

$$(g_\pm\Gamma\Psi) \equiv \int_{-1}^{1}\int_{-1}^{1} g_\pm(x)K(x-t)\Psi(t)\,\mathrm{d}t\,\mathrm{d}x \tag{56}$$

by two different methods, the first by applying the operator Γ on $g_\pm$ and the second by applying it on Ψ. In such a manner, one obtains

$$\int_{-1}^{1} \Psi(t)\,\mathrm{e}^{\pm\theta t}\mathrm{d}t = \int_{-1}^{1} g_\pm(t)(\mathrm{e}^{-z\theta t} + a_+\mathrm{e}^{\theta t} + a_-\mathrm{e}^{-\theta t})\,\mathrm{d}t. \tag{57}$$

Introduce the Laplace transform of functions $g_{c,s}$:

$$\hat{G}_c(p) = \int_{-1}^{1} g_c(t)\,\mathrm{e}^{p\theta t}\,\mathrm{d}t, \qquad \hat{G}_s(p) = \int_{-1}^{1} g_s(t)\,\mathrm{e}^{p\theta t}\,\mathrm{d}t. \tag{58}$$

Due to the symmetry properties of $g_{c,s}$, $\hat{G}_c(-p) = \hat{G}_c(p)$ and $\hat{G}_s(-p) = -\hat{G}_s(p)$.

Using (58), we obtain

$$\int_{-1}^{1} \Psi(t)\,\mathrm{e}^{\theta t}\,\mathrm{d}t = G(-z) + a_+G(1) + a_-G(-1), \tag{59}$$

$$\int_{-1}^{1} \Psi(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t = G(z) + a_+G(-1) + a_-G(1), \tag{60}$$

where $G(p) = \hat{G}_c(p) + \hat{G}_s(p)$.

On the other hand, multiplying (55) by $e^{\pm\theta t}$, integrating the results from -1 to 1 and taking into account that $\Psi(\pm 1) = 0$, one finds

$$(z-1)\int_{-1}^{1} e^{\theta t}\Psi(t)\,dt = (z+1)a_+G(1) + (z-1)a_-G(-1), \tag{61}$$

$$(z+1)\int_{-1}^{1} e^{-\theta t}\Psi(t)\,dt = (z+1)a_+G(-1) + (z-1)a_-G(1). \tag{62}$$

These and previous equations permits us to calculate constants $a_\pm$:

$$a_\pm = -\frac{(1\mp z)G(\mp z)}{2G(1)}. \tag{63}$$

Combining the above formulae, one finds that (55) takes the form

$$\theta^{-1}\Psi'(t) + z\Psi(t) = \frac{z^2-1}{G(1)}[\hat{G}_c(z)g_s(t) - \hat{G}_s(z)g_c(t)] \tag{64}$$

whose solution obeying $\Psi(\pm 1) = 0$ is

$$\Psi(t) = \frac{(z^2-1)\theta}{G(1)}\, e^{-z\theta t}\left[\hat{G}_c(z)\int_{-1}^{t} g_s(y)\, e^{z\theta y}\,dy - \hat{G}_s(z)\int_{-1}^{t} g_s(y)\, e^{z\theta y}\,dy\right]. \tag{65}$$

The above formulae relate the general plane-wave solution (48) to two special cases corresponding to $z = \pm 1$. Such formulae are called the embedding formulae in the theory of diffraction and can be derived in more general settings (see, e.g., [17] and references therein).

5. Positivity relations

The method discussed in the previous sections works only if the main equation (1) has a unique solution, i.e. the only solution for $f(x) = 0$ is $g(t) = 0$. This uniqueness follows from the fact that the Fourier transform of the kernel $K(w)$ (2) is strictly positive for all p and $\nu > -1/2$ (see (C.13)). It means that the function $K(w)$ is a positive-definite function and the double integral

$$\int_{-1}^{1} dt \int_{-1}^{1} g(t)K(|x-t|)g(x)\,dx \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-1}^{1} g(t)\, e^{ipt}\,dt\right|^2 \hat{K}(p)\,dp \tag{66}$$

is positive for any function $g(x)$ nonzero in the interval $[-1, 1]$.

It is instructive to check how this property reflects in the general solution. Equation (65) represents the solution of (48). As for usual diffraction problems, it is physically clear (and can be proved, e.g., by using the series expansions as in appendix A) that such a solution exist for all $\theta > 0$ and any z but (65) contains $G(1)$ in the numerator, and in order that the solution remains finite, it is necessary that this quantity is always nonzero.

By construction, $G(1)$ is the Laplace transform of the sum $g_c + g_s$:

$$G(1) = \int_{-1}^{1} (g_c(t) + g_s(t))\, e^{\theta t}\,dt = \int_{-1}^{1} g_c(t)\cosh(\theta t)\,dt + \int_{-1}^{1} g_s(t)\sinh(\theta t)\,dt. \tag{67}$$

Here, $g_c(t)$ and $g_s(t)$ are solutions of (8) thus

$$\cosh(\theta x) = (\Gamma g_c)(x), \quad \sinh(\theta x) = (\Gamma g_s)(x) \tag{68}$$

and

$$G(1) = \int_{-1}^{1} dt \int_{-1}^{1} g_c(t)K(|x-t|)g_c(x)\,dx + \int_{-1}^{1} dt\int_{-1}^{1} g_s(t)K(|x-t|)g_s(x)\,dx. \tag{69}$$

Due to the positiveness of the kernel $K(w)$ (66) for all $\theta > 0$

$$G(1) > 0 \tag{70}$$

and the solution (65) is finite for all z.

The same positivity condition permits also to establish the positivity and finiteness of $\eta(\theta)$. Formally, this function is defined as the limit, when $t \to 1$, of the ratio of two solutions (25) and it is not obvious that it remains bounded for all $\theta > 0$.

Let us define two functions

$$\hat{F}_c(p) = \int_{-1}^{1} t g_c(t)\, \mathrm{e}^{p\theta t}\, \mathrm{d}t, \qquad \hat{F}_s(p) = \int_{-1}^{1} t g_s(t)\, \mathrm{e}^{p\theta t}\, \mathrm{d}t, \tag{71}$$

which differ from (58) by the factor t in the integrand.

Multiplying (28) by $\mathrm{e}^{z\theta t}$ and integrating the both parts over the interval $[-1, 1]$, one finds

$$\left[\frac{\nu - 1/2 - \rho}{\theta} + \frac{1}{\eta}\right] \hat{G}_c(z) + \frac{z}{\eta} \hat{G}_s(z) = z\hat{F}_c(z) + \hat{F}_s(z),$$
$$\left[\frac{\nu - 1/2 + \rho}{\theta} + \eta\right] \hat{G}_s(z) + z\eta \hat{G}_c(z) = \hat{F}_c(z) + z\hat{F}_s(z). \tag{72}$$

From (39) and (40), it follows that

$$\frac{\partial}{\partial\theta}\hat{G}_c(z) - \frac{1/2 + \rho}{\theta}\hat{G}_c(z) = \hat{F}_s(z) + z\hat{F}_c(z),$$
$$\frac{\partial}{\partial\theta}\hat{G}_s(z) - \frac{1/2 - \rho}{\theta}\hat{G}_s(z) = \hat{F}_c(z) + z\hat{F}_s(z). \tag{73}$$

Combining these relations, one concludes that

$$\frac{\partial}{\partial\theta}\hat{G}_c(z) - \left[\frac{\nu}{\theta} + \frac{1}{\eta}\right]\hat{G}_c(z) = \frac{z}{\eta}\hat{G}_s(z), \qquad \frac{\partial}{\partial\theta}\hat{G}_s(z) - \left[\frac{\nu}{\theta} + \eta\right]\hat{G}_s(z) = z\eta\hat{G}_c(z). \tag{74}$$

Put in the last expressions $z = 1$. Then, $G(1) = \hat{G}_c(1) + \hat{G}_s(1)$ obeys the equation

$$\frac{\partial}{\partial\theta}G(1) = \left[\frac{\nu}{\theta} + \eta + \frac{1}{\eta}\right]G(1) \tag{75}$$

whose solution is

$$G(1) = C'(\nu)\,\theta^{\nu}\, \exp\int^{\theta} (\eta(\theta') + \eta^{-1}(\theta'))\,\mathrm{d}\theta', \tag{76}$$

where $C'(\nu)$ is a constant.

Using (24), (39) and (40) with $t = 1$, one concludes that

$$\frac{\partial}{\partial\theta}k_c = \frac{1/2 + \rho}{\theta}k_c + k_c\eta. \tag{77}$$

Solving it and using (42), we find

$$k_c^2 = C''(\nu)\frac{\theta}{\eta}\exp\int^{\theta} (\eta(\theta') + \eta^{-1}(\theta'))\,\mathrm{d}\theta' \tag{78}$$

with another constant $C''(\nu)$.

Comparison of this expression with (76) proves that

$$\eta\, k_c^2 = C(\nu)\,\theta^{1-\nu}G(1). \tag{79}$$

To find the constant of proportionality $C(\nu)$, we compare the both sides of this equation when $\theta \to 0$.

From appendix C, it follows that for $|\nu| < 1/2$ and $\theta \to 0$ the dominant contribution to $G(1)$ is due to $\hat{G}_c(1)$ and it is straightforward to check that

$$C(\nu) = \frac{\cos\pi\nu}{2^{\nu+1}\pi^{3/2}\Gamma(1/2 - \nu)}. \tag{80}$$

For $|\nu| < 1/2$, $C(\nu) > 0$ and due to the positivity of $G(1)$, it follows from (79) that $\eta > 0$ for all positive $\theta > 0$. As the only possible moving singularities of η are poles, this reasoning proves that η is bounded for all $\theta > 0$ (the behaviour of η at large θ is fixed by (C.26)).

The list of solutions of the Painlevé III equation (44), which remain bounded as $\theta \to \infty$ along the real axis has been presented in [13]. Such solutions form a family parametrized by one parameter λ, which determined the large θ behaviour of solutions

$$\eta(\theta) \underset{\theta\to\infty}{\longrightarrow} 1 - \lambda\Gamma(\nu + 1/2)2^{-2\nu}\theta^{-\nu-1/2}e^{-2\theta}. \tag{81}$$

It has been proved in [13] that bounded solutions are characterized by the following behaviour at small θ:

$$\eta(\theta) \underset{\theta\to 0}{\longrightarrow} (2\theta)^{\sigma}B + (2\theta)B_1 + (2\theta)^{1+2\sigma}B_2 + (2\theta)^{2-\sigma}B_3 + O(\theta^{2+\sigma}), \tag{82}$$

where σ (restricted by inequalities $-1 < \operatorname{Re}\sigma < 1$) is related with λ as follows:

$$\sigma = \frac{2}{\pi}\arcsin(\pi\lambda). \tag{83}$$

Coefficients B_j are

$$B_1 = -\frac{\nu}{(1-\sigma)^2}, \quad B_2 = B^2\frac{\nu}{(1+\sigma)^2}, \quad B_3 = \frac{1}{16B(1-\sigma)^4}(4\nu^2 - (1-\sigma)^2) \tag{84}$$

with $B = B(\sigma, \nu)$ being a function of σ and ν:

$$B(\sigma,\nu) = 2^{-3\sigma}\frac{\Gamma^2((1-\sigma)/2)\Gamma((1+\sigma)/2+\nu)}{\Gamma^2((1+\sigma)/2)\Gamma((1-\sigma)/2+\nu)}. \tag{85}$$

These bounded solutions have been represented in [13] as infinite series of multiple integrals and the proof that they obey the Painlevé III equation (44) is quite complicated.

The solution of the integral equation (1) discussed above gives rise to the function $\eta(\theta)$ (25), which, as have been proved, is a certain bounded solution of the same Painlevé III equation (44). It corresponds to a particular case of general theory of [13] with $|\nu| < 1/2$ and parameter λ taking a special value (cf. (C.10), (85), and (C.26)):

$$\lambda = \frac{\cos(\pi\nu)}{\pi} \tag{86}$$

and

$$\sigma = 1 - 2\nu. \tag{87}$$

For such σ, the term B_3 in (84) is identically zero and the asymptotics (C.9) represents the two leading terms when $\theta \to 0$.

It seems that the case $-1/2 < \nu < 0$ is not fully cover by the analysis of [13] as $\sigma = 1-2\nu$ is beyond the interval $[-1, 1]$ assumed in [13], $1 < \sigma < 2$. Although in this case the dominant behaviour at small θ is linear in θ, $\eta \to -\theta/(2\nu)$, its asymptotics at large θ is given by the same formula (81) but with a value of λ (86) obtained by the inversion of (83).

6. Summary

We demonstrate that special solutions of the integral equation (1) can be obtained from ODEs (26), (27) or (39), (40) with entering constants calculated using the Painlevé III equation (44) with the known asymptotics at small and large arguments.

More precisely, to solve the equation

$$\int_{-1}^{1} |x-t|^{\nu}K_{\nu}(\theta|x-t|)g(t)\,dt = e^{-\theta zx}, \tag{88}$$

one has to perform the following steps.

- Solve the Painlevé III equation

$$\frac{d^2\eta}{d\theta^2} = \eta^{-1}\left(\frac{d\eta}{d\theta}\right)^2 - \theta^{-1}\frac{d\eta}{d\theta} - \frac{2\nu(1-\eta^2)}{\theta} + \eta^3 - \frac{1}{\eta} \tag{89}$$

for $\eta(\theta)$ starting from the asymptotics (C.9) at small θ.

- The knowledge of $\eta(\theta)$ permits to calculate function $\rho(\theta)$

$$\rho = \frac{\theta}{2\eta}(1 - \eta' - \eta^2). \tag{90}$$

- Solve ODEs for the functions $g_c(t)$ and $g_s(t)$:

$$(1-t^2)g_c'(t) = \left(\nu + \frac{1}{2} - \rho\right) t g_c(t) + \left(\frac{\nu + t\frac{1}{2} + \rho}{\eta} + \theta(1-t^2)\right) g_s(t),$$
$$(1-t^2)g_s'(t) = \left(\nu + \frac{1}{2} + \rho\right) t g_s(t) + \left(\eta\left(\nu + t\frac{1}{2} - \rho\right) + \theta(1-t^2)\right) g_c(t). \tag{91}$$

These equations have to be completed by initial conditions. By symmetry, $g_s(0) \equiv 0$ and $g_0 \equiv g_c(0)$ can be calculated from (39) with $t = 0$ and $g_s = 0$:

$$\frac{dg_0}{d\theta} = \frac{1/2 + \rho}{\theta} g_0, \tag{92}$$

whose solution with the asymptotics (C.5) may be written as

$$g_0 = \frac{2^\nu \cos(\pi\nu)}{\pi\Gamma(1-\nu)} \frac{\theta^{1-\nu}}{\eta} \exp\left(\int_0^\theta \left[\frac{2\nu - 1}{\theta'} + \eta^{-1}(\theta') - \eta(\theta') + \frac{\eta'}{\eta}(\theta')\right] d\theta'\right). \tag{93}$$

(The terms in the exponent are such that the integral tends to zero, when $\theta \to 0$ for all $|\nu| < 1/2$.)

- When functions $g_c(t)$ and $g_s(t)$ are calculated, one can find $\Psi(t)$ from (65) and, finally, the solution $g(t)$ from (50)

$$g(t) = \Psi(t) - a_+ g_+(t) - a_- g_-(t), \tag{94}$$

where all necessary quantities are defined in section 4.

For large θ, it may be more convenient to perform the same steps but starting not with asymptotics at $\theta \to 0$ but with limiting behaviours at $\theta \to \infty$. The necessary formulae are presented in appendix C.

This solution is quite tedious but it is exact, explicit and requires only the calculation of a few ODEs with the known initial conditions.

Our finding generalizes the results obtained in [6] for the special case $\nu = 0$. An interesting consequence of our investigation is expression (47) for a solution of the Painlevé III equation with asymptotics given by (C.9) and (C.26) as the ratio of two infinite series of spheroidal functions with the known coefficients. The positive definiteness of the kernel forces this Painlevé III solution to be bounded for all positive arguments, thus giving another proof of connection formulae for the Painlevé III equation [13] in a special case.

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Appendix A. Series solution of the integral equation

Let μ_m and $Y_m(\gamma)$ be generalized eigenvalues and eigenfunctions of operator (2):

$$\mu_m \int_0^\pi (\sin\gamma)^{-2\nu} K(|\cos\beta - \cos\gamma|) Y_m(\gamma)\,\mathrm{d}\gamma = Y_m(\beta). \tag{A.1}$$

As the symmetrized version of the kernel has the form

$$\tilde{K}(\beta,\gamma) = (\sin\beta)^{-\nu} K(|\cos\beta - \cos\gamma|)(\sin\gamma)^{-\nu}, \tag{A.2}$$

the eigenfunctions $Y_m(\gamma)$ form an orthogonal system of functions

$$\int_0^\pi (\sin\gamma)^{-2\nu} Y_m(\gamma) Y_n(\gamma)\,\mathrm{d}\gamma = N_m \delta_{mn}, \tag{A.3}$$

where N_m is the normalization constant.

If a function $f(x)$ is expanded into a series of functions $Y_m(\gamma)$

$$f(\cos\gamma) = \sum_m a_m Y_m(\gamma), \tag{A.4}$$

then the formal solution of the integral equation (1) is

$$g(\cos\gamma) = (\sin\gamma)^{-2\nu-1} \sum_{m=1}^{\infty} \mu_m a_m Y_m(\gamma). \tag{A.5}$$

To find explicitly eigenvalues and eigenfunctions of (A.1), it was noted in [1] that the function $K(\sqrt{x^2+y^2})$ plays the role of the Green function of the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{2\nu}{y}\frac{\partial}{\partial y} - \theta^2\right)\Psi(x,y) = 0. \tag{A.6}$$

Therefore, the function

$$\Psi^{(\mathrm{ref})}(x,y) = \int_{-1}^{1} g(t) K(\sqrt{(t-x)^2+y^2})\,\mathrm{d}t \tag{A.7}$$

is a univalued solution of (A.6) in all points of the (x, y)-plane except the strip $[-1, 1]$, which exponentially decays at large distances

$$\Psi^{(\mathrm{ref})}(r\cos\phi,\ r\sin\phi) \underset{r\to\infty}{\longrightarrow} \sqrt{\frac{2}{\theta\pi}}\, r^{\nu-1/2}\,\mathrm{e}^{-\theta r} \int_{-1}^{1} g(t)\,\mathrm{e}^{\theta t\cos\phi}\,\mathrm{d}t. \tag{A.8}$$

Consider a Dirichlet-type problem of finding the solution of (A.6) in the form

$$\Psi(x,y) = \Psi^{(\mathrm{inc})}(x,y) + \Psi^{(\mathrm{ref})}(x,y), \tag{A.9}$$

where $\Psi^{(\mathrm{inc})}(x,y)$ also obeys (A.6) and the total field $\Psi(x,0)$ is zero at the interval $[-1,1]$:

$$\Psi(x,0) = 0, \qquad -1 \leqslant x \leqslant 1. \tag{A.10}$$

It means that the unknown function $g(t)$ obeys (1) with $f(x) = -\Psi^{(\mathrm{inc})}(x,0)$.

Equation (A.6) permits the separation of variables in the elliptic coordinates

$$x = \cosh\xi\cos\gamma, \qquad y = \sinh\xi\sin\gamma. \tag{A.11}$$

It is plain that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{2}{\cosh 2\xi - \cos 2\gamma}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\gamma^2}\right) \tag{A.12}$$

and

$$\frac{\partial}{\partial y} = \frac{2}{\cosh 2\xi - \cos 2\gamma}\left(\cosh\xi\sin\gamma\frac{\partial}{\partial\xi} + \sinh\xi\cos\gamma\frac{\partial}{\partial\gamma}\right). \tag{A.13}$$

If $\Psi(x,y) = X(\xi)Y(\gamma)$, then $Y(\gamma)$ and $X(\xi)$ obey

$$Y_m'' - 2\nu \cot\gamma\, Y_m' - \left(\alpha_m - \tfrac{1}{2}\theta^2 \cos 2\gamma\right) Y_m = 0 \tag{A.14}$$

and

$$X_m'' - 2\nu \coth\xi\, X_m' + \left(\alpha_m - \tfrac{1}{2}\theta^2 \cosh 2\xi\right) X_m = 0. \tag{A.15}$$

The separation constant α_m is chosen to ensure the symmetry properties of solutions

$$Y_m(-\gamma) = Y_m(\gamma), \qquad Y_m(\gamma) = Y_m(\gamma+\pi). \tag{A.16}$$

The solution $X_m(\xi)$ is fixed by its normalization at infinity:

$$X_m(\xi) \underset{\xi\to\infty}{\longrightarrow} C\mathrm{e}^{\xi(\nu-1/2)} \exp\left(-\frac{\theta}{2}\,\mathrm{e}^{\xi}\right), \tag{A.17}$$

where C is a constant.

The above equations are the particular cases of spheroidal equation [12]. $Y_m(\gamma)$ is an even periodic angular solution of (A.14), and $X_m(\xi)$ is the radial solution of the modified equation (A.15) decreasing at the infinity. It is known that Y_m are the orthogonal functions on the interval $[0,\ \pi]$ as in (A.3) and they form a complete set in the space of even functions on this interval.

Therefore, the reflected univalued field in the (x, y)-plane cut along the interval $[-1, 1]$, decaying at infinity (cf. (A.17)) and obeying (A.10) with $f(x)$ as in (A.4), can be represented as a formal series

$$\Psi^{(\mathrm{ref})}(\xi,\gamma) = \sum_m a_m \frac{X_m(\xi)}{X_m(0)} Y_m(\gamma). \tag{A.18}$$

As for the usual Dirichlet problem, the value of $g(t)$ in (A.7) is related with the normal derivative of $\Psi^{(\mathrm{ref})}$ at the strip $[-1, 1]$. At small w

$$K_\nu(w) \underset{w\to 0}{\longrightarrow} w^{-\nu}\frac{\Gamma(\nu)}{2^{1-\nu}} + w^{\nu}\frac{\Gamma(-\nu)}{2^{1+\nu}}. \tag{A.19}$$

As $w = \sqrt{(x-t)^2 + y^2}$ and for $\nu < 1/2$

$$\lim_{y\to 0+} \frac{y^{1-2\nu}}{(x^2+y^2)^{1-\nu}} = \frac{\sqrt{\pi}\,\Gamma(1/2-\nu)}{\Gamma(1-\nu)}\delta(x), \tag{A.20}$$

one obtains that [1]

$$g(x) = -\frac{2^{\nu}\theta^{-\nu}}{\sqrt{\pi}\,\Gamma(1/2-\nu)} \lim_{y\to 0} y^{-2\nu}\frac{\partial}{\partial y}\Psi^{(\mathrm{ref})}(x,y). \tag{A.21}$$

The behaviour of the decaying solution $X_m(\xi)$ at small ξ follows from (A.15):

$$X_m(\xi) \underset{\xi\to 0}{\longrightarrow} A_m + B_m \xi^{1+2\nu}. \tag{A.22}$$

For x in the strip $[-1, 1]$ and small y, $y = \xi \sin\gamma$. Therefore,

$$g(\cos\gamma) = -\frac{2^{\nu}(2\nu+1)\theta^{-\nu}}{\sqrt{\pi}\,\Gamma(1/2-\nu)} (\sin\gamma)^{-2\nu-1} \sum_m a_m \frac{B_m}{A_m} Y_m(\gamma). \tag{A.23}$$

It means that the eigenfunctions of operator (A.1) equal Y_m and the corresponding eigenvalues are

$$\mu_m = -\frac{2^{\nu}\theta^{-\nu}(2\nu+1)B_m}{\sqrt{\pi}\,\Gamma(1/2-\nu)A_m}, \tag{A.24}$$

where A_m and B_m are determined from the dominant behaviour at small ξ of the decaying solution of the radial equation $X_m(\xi)$ (A.22).

Appendix B. Spheroidal functions

A convenient method to find periodic functions $Y_m(\gamma)$ is to expand them into a series of Gegenbauer's polynomials [12]:

$$Y_m(\gamma) = \sum_{n\equiv m \bmod 2} b_n C_n^{-\nu}(\cos\gamma), \tag{B.1}$$

where the summation is performed over all non-negative integers of the same parity as m.

Here, $C_n^{\mu}(x)$ are the polynomial solutions of the equation

$$(1-x^2)y'' - (2\mu+1)xy' + n(n+2\mu)y = 0 \tag{B.2}$$

and $C_n^{\mu}(\cos\gamma)$ are the solutions of

$$y'' + 2\mu\cot\gamma\, y' + n(n+2\mu)y = 0. \tag{B.3}$$

We impose the standard normalization [14]

$$C_n^{\mu}(1) = \frac{\Gamma(n+2\mu)}{n!\Gamma(2\mu)} \tag{B.4}$$

so the orthogonality relation for Gegenbauer's polynomials is

$$\int_{-1}^{1}(1-x^2)^{\mu-1/2}C_n^{\mu}(x)C_m^{\mu}(x)\,\mathrm{d}x = \delta_{nm}\frac{2^{1-2\mu}\pi\Gamma(n+2\mu)}{n![\Gamma(\mu)]^2(n+\mu)}. \tag{B.5}$$

Note that [14]

$$\mu C_n^{\mu}(\cos\gamma) \underset{\mu\to 0}{\longrightarrow} \frac{2}{n}\cos(n\gamma), \tag{B.6}$$

so for $\mu\to 0$, series (B.1) can be transformed to series used for even Mathieu functions.

Using the recurrence relation for Gegenbauer's polynomials [14]

$$(n+1)C_{n+1}^{\mu}(x) = 2(n+\mu)xC_n^{\mu}(x) - (n+2\mu-1)C_{n-1}^{\mu}(x) \tag{B.7}$$

gives

$$\cos 2\gamma C_n^{-\nu}(\cos\gamma) = A_n C_n^{-\nu}(\cos\gamma) + B_{n-2}C_{n-2}^{-\nu}(\cos\gamma) + D_{n+2}C_{n+2}^{-\nu}(\cos\gamma), \tag{B.8}$$

where

$$A_n = -\frac{\nu(1+\nu)}{(n-\nu)^2-1},\quad B_n = \frac{(n-2\nu+1)(n-2\nu)}{2(n-\nu+2)(n-\nu+1)},\quad D_n = \frac{(n-1)n}{2(n-\nu-2)(n-\nu-1)}. \tag{B.9}$$

Substituting the formal series (B.1) into (A.14) leads to a three-diagonal matrix for the determination of b_n with the fixed parity

$$\left[\tfrac{1}{2}\theta^2 A_n - n(n-2\nu)\right]b_n + \tfrac{1}{2}\theta^2 B_n b_{n+2} + \tfrac{1}{2}\theta^2 D_n b_{n-2} = \alpha_m b_n. \tag{B.10}$$

The eigenvalues of the above matrix determine separation constants α_m.

From (B.5), it follows that the normalization constant for this solution is

$$N_m = \sum_{n\equiv m \bmod 2} b_n^2 \frac{2^{1-2\mu}\pi\Gamma(n+2\mu)}{n![\Gamma(\mu)]^2(n+\mu)}. \tag{B.11}$$

The function $\mathrm{e}^{\theta x}$ is a solution of (A.6). Therefore, it can be expanded in elliptic coordinates (A.11) as

$$\mathrm{e}^{\theta\cosh\xi\cos\gamma} = \sum_m Y_m(\gamma)\tilde{X}_m(\xi), \tag{B.12}$$

where $\tilde{X}_m(\xi)$ are the certain solutions (to be discussed below) of the modified equation (A.15).

Using (A.3), one obtains

$$\tilde{X}_m(\xi) = \frac{1}{N_m}\int_0^{\pi} (\sin\gamma)^{-2\nu}\, \mathrm{e}^{\theta\cosh\xi\cos\gamma} Y_m(\gamma)\,\mathrm{d}\gamma. \tag{B.13}$$

Due to the symmetry properties (A.16)

$$\tilde{X}_{2m}(\xi) = \frac{1}{N_{2m}}\int_0^{\pi} (\sin\gamma)^{-2\nu} \cosh[\theta\cosh\xi\cos\gamma] Y_{2m}(\gamma)\,\mathrm{d}\gamma \tag{B.14}$$

and

$$\tilde{X}_{2m+1}(\xi) = \frac{1}{N_{2m+1}}\int_0^{\pi} (\sin\gamma)^{-2\nu} \sinh[\theta\cosh\xi\cos\gamma] Y_{2m+1}(\gamma)\,\mathrm{d}\gamma. \tag{B.15}$$

Using Gegenbauer's integral [14]

$$n!\int_0^{\pi} \mathrm{e}^{\mathrm{i}z\cos\gamma} C_n^{\mu}(\cos\gamma)(\sin\gamma)^{2\mu}\mathrm{d}\gamma = 2^{\mu}\sqrt{\pi}\,\Gamma(\mu+1/2)\Gamma(n+2\mu)\mathrm{i}^n z^{-\mu} J_{n+\mu}(z), \tag{B.16}$$

one obtains

$$\tilde{X}_m(\xi) = \frac{2^{-\nu}\Gamma(1/2-\nu)}{N_m\Gamma(-2\nu)}\sum_{n\equiv m \bmod 2} b_n \frac{\Gamma(n-2\nu)}{n!}(\theta\cosh(\xi))^{\nu} I_{n-\nu}(\theta\cosh(\xi)), \tag{B.17}$$

where $I_{\mu}(x)$ are the modified Bessel functions of the first kind.

These functions increase exponentially at large ξ and the series (B.17) represent a solution of (A.15) of the first kind. As $I_{n-\nu}$ and $(-1)^n K_{n-\nu}$ obey the same recurrent relations, the exponentially decaying solutions called the solutions of the third kind (denoted in appendix A by $X_m(\xi)$) take the form

$$X_m(\xi) = \frac{2^{-\nu}\Gamma(1/2-\nu)}{N_m\Gamma(-2\nu)}\sum_{n\equiv m \bmod 2} (-1)^n b_n \frac{\Gamma(n-2\nu)}{n!}(\theta\cosh(\xi))^{\nu} K_{n-\nu}(\theta\cosh(\xi)). \tag{B.18}$$

Appendix C. Limiting behaviour

The purpose of this appendix is to determine the behaviour of $g_c(t)$ and $g_s(t)$ defined in (8) in the limit of small θ and large θ.

C.1. Small θ behaviour

From (5), it follows that at small θ

$$K(w) \underset{\theta\to 0}{\longrightarrow} \theta^{-\nu}\frac{\Gamma(\nu)}{2^{1-\nu}} + w^{2\nu}\theta^{\nu}\frac{\Gamma(-\nu)}{2^{1+\nu}}. \tag{C.1}$$

Therefore, to find the necessary solutions of (8) when $\theta\to 0$, it is necessary to first solve the equation

$$\int_{-1}^{1} |x-t|^{2\nu} g(t)\,\mathrm{d}t = f(x) \tag{C.2}$$

when the function $f(x)$ equals 1 for x.

Although the general solution of such equation is known [8], [15], solutions with $f(x)=1$ and $f(x)=x$ can easily be obtained by the direct application of Latta's method [5]. After a simple algebra, one finds that the solutions of equations

$$\int_{-1}^{1} |x-t|^{2\nu} g_0(t)\,\mathrm{d}t = 1, \qquad \int_{-1}^{1} |x-t|^{2\nu} g_1(t)\,\mathrm{d}t = x \tag{C.3}$$

are

$$g_0(t) = \frac{\cos(\pi\nu)}{\pi(1-t^2)^{\nu+1/2}}, \qquad g_1(t) = -\frac{\cos(\pi\nu)t}{2\pi\nu(1-t^2)^{\nu+1/2}}. \tag{C.4}$$

From (C.1), we obtain that

$$g_c(t) \underset{\theta\to 0}{\longrightarrow} \mu_c g_0(t), \qquad g_s(t) \underset{\theta\to 0}{\longrightarrow} \mu_s g_1(t), \tag{C.5}$$

where the constants μ_c and μ_s are determined from the conditions

$$\mu_c\left(\theta^\nu \frac{\Gamma(-\nu)}{2^{1+\nu}} + \theta^{-\nu}\frac{\cos(\pi\nu)\Gamma(1/2-\nu)\Gamma(\nu)}{\sqrt{\pi}2^{1-\nu}\Gamma(1-\nu)}\right) = 1, \qquad \mu_s\theta^\nu\frac{\Gamma(-\nu)}{2^{1+\nu}} = \theta. \tag{C.6}$$

From these relations, it follows that

$$\eta(\theta) \underset{\theta\to 0}{\longrightarrow} -\frac{\mu_s}{2\nu\mu_c} = -\theta^{1-2\nu}\frac{\cos(\pi\nu)\Gamma(1/2-\nu)\Gamma(\nu)}{\sqrt{\pi}2^{1-2\nu}\nu\Gamma(1-\nu)\Gamma(-\nu)} - \frac{\theta}{2\nu}. \tag{C.7}$$

Using the standard formulae for the Γ-function

$$\Gamma(2\nu) = \frac{2^{2\nu-1}}{\sqrt{\pi}}\Gamma(\nu)\Gamma(\nu+1/2), \qquad \Gamma(1/2+\nu)\Gamma(1/2-\nu) = \frac{\pi}{\cos(\pi\nu)}, \tag{C.8}$$

one obtains that the limiting behaviour of $\eta(\theta)$ at small θ is

$$\eta(\theta) \underset{\theta\to 0}{\longrightarrow} B(2\theta)^{1-2\nu} - \frac{\theta}{2\nu} = \begin{cases} B(2\theta)^{1-2\nu}, & 0<\nu<1/2,\\ -\theta/(2\nu), & -1/2<\nu<0,\end{cases} \tag{C.9}$$

where

$$B = 2^{-3(1-2\nu)}\frac{\Gamma^2(\nu)}{\Gamma^2(1-\nu)\Gamma(2\nu)}. \tag{C.10}$$

C.2. Large θ behaviour

To find the behaviour of solutions when $\theta \to \infty$, it is convenient to consider instead of functions $g_{c,s}$ (8) a solution $g_-(v)$ corresponding to the integral equation

$$\int_{-1}^{1} |u-v|^\nu K_\nu(\theta|u-v|)g_-(v)\,\mathrm{d}v = \mathrm{e}^{-\theta u}. \tag{C.11}$$

To find the asymptotic behaviour of g_- for large θ, we first consider the equation

$$\int_0^\infty |u-v|^\nu K_\nu(|u-v|)g_0(v)\,\mathrm{d}v = \mathrm{e}^{-u}. \tag{C.12}$$

Its solution can be calculated either by the Wiener–Hopf method [16] using the known Fourier transform of the kernel

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}px}K_\nu(\theta x)x^\nu\,\mathrm{d}x = \frac{(2\theta)^\nu\sqrt{\pi}\Gamma(\nu+1/2)}{(p^2+\theta^2)^{\nu+1/2}} \tag{C.13}$$

or by the expansion formulae analogous to the ones described in appendix A but for half-line integration [18]

$$n!\int_0^\infty |x-t|^\nu K_\nu\left(\frac{1}{2}|x-t|\right)t^{-\nu-1/2}\mathrm{e}^{-t/2}L_n^{-\nu-1/2}(t)\,\mathrm{d}t = \sqrt{\pi}\Gamma(\nu+1/2)\Gamma(n+1/2-\nu)\,\mathrm{e}^{-x/2}L_n^{-\nu-1/2}(x), \tag{C.14}$$

where $L_n^\lambda(x)$ with $n = 0, 1, \ldots$ are the Laguerre polynomials.

One obtains

$$g_0(v) = C_\nu v^{-\nu-1/2}\mathrm{e}^{-v}, \qquad C_\nu = \frac{\sqrt{2}}{\pi^{3/2}}\cos\pi\nu. \tag{C.15}$$

Using this solution, it is easy to show that the dominant approximation for the solution of (C.11) when $\theta \to \infty$ has the form

$$g_-^{(0)}(t) = C_\nu \sqrt{\theta}(1+t)^{-\nu-1/2}\mathrm{e}^{-t\theta}. \tag{C.16}$$

Straightforward transformations give

$$\int_{-1}^{1} |x-t|^\nu K_\nu(\theta|x-t|) g_-^{(0)}(t)\,\mathrm{d}t = F_1(x) - F_2(x), \tag{C.17}$$

where

$$F_1(x) = \int_{-1}^{\infty} |x-t|^\nu K_\nu(\theta|x-t|) g_-^{(0)}(t)\,\mathrm{d}t, \quad F_2(x) = \int_{1}^{\infty} |x-t|^\nu K_\nu(\theta|x-t|) g_-^{(0)}(t)\,\mathrm{d}t. \tag{C.18}$$

Changing the variable in the first integral as $t = -1 + v/\theta$ and using (C.12) leads to

$$F_1(x) = \mathrm{e}^{-\theta x}. \tag{C.19}$$

In the integral for $F_2(x)$, it is convenient to put $t = 1 + s/\theta$. In this way, one obtains

$$F_2(x) = \theta^{-\nu-1/2} C_\nu \int_0^\infty \left(2 + \frac{s}{\theta}\right)^{-\nu-1/2} (s + \theta(1-x))^\nu K_\nu(s + \theta(1-x))\,\mathrm{e}^{-\theta-s}\,\mathrm{d}s. \tag{C.20}$$

When θ is large, one can drop the term s/θ in the first bracket and use the asymptotic form of the K_ν function (4)

$$F_2(x) \underset{\theta\to\infty}{\longrightarrow} \theta^{-\nu-1/2} 2^{-\nu-1}\sqrt{\pi}\mathrm{e}^{-2\theta+\theta x} C_\nu \int_0^\infty (s + \theta(1-x))^{\nu-1/2}\mathrm{e}^{-2s}\,\mathrm{d}s. \tag{C.21}$$

This formula means that close to the right-hand side, $x = 1$, the additional contribution related with the integration in finite limits is

$$F_2(x) \approx \theta^{-\nu-1/2} 2^{-\nu-1}\sqrt{\pi}\mathrm{e}^{-2\theta+\theta x} C_\nu \int_0^\infty s^{\nu-1/2}\mathrm{e}^{-2s}\mathrm{d}s = \mathrm{e}^{\theta x}\delta, \tag{C.22}$$

where

$$\delta = \frac{1}{2\pi}\theta^{-\nu-1/2} 2^{-2\nu} \cos(\pi\nu)\,\mathrm{e}^{-2\theta}\Gamma(\nu+1/2). \tag{C.23}$$

Using (C.11), it is obvious that to cancel contribution $F_2(x)$, it is necessary to modify the approximation (C.16) as follows:

$$g_-(t) \underset{\theta\to\infty}{\longrightarrow} C_\nu\sqrt{\theta}\left[(1+t)^{-\nu-1/2}\mathrm{e}^{-t\theta} + \delta(1-t)^{-\nu-1/2}\mathrm{e}^{t\theta}\right]. \tag{C.24}$$

This function consists of two terms. The first dominates close to $t = -1$ and the second near $t = 1$. According to (25), the function $\eta(\theta)$ is determined from the limit

$$\eta(\theta) = \lim_{t\to 1} \frac{g_-(-t) - g_-(t)}{g_-(-t) + g_-(t)}. \tag{C.25}$$

Using (C.24), we find that

$$\eta(\theta) \underset{\theta\to\infty}{\longrightarrow} 1 - \frac{\cos(\pi\nu)}{\pi}\Gamma(\nu+1/2) 2^{-2\nu}\theta^{-\nu-1/2}\mathrm{e}^{-2\theta}. \tag{C.26}$$

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