Diffraction and Painlevé Equation

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Abstract : We have investigated exact solutions of two dimensional diffraction problems for various geometries. Utilising differential and integral equations formalism, we have succeeded in representing solutions of Painlevé III equation as a ratio of two infinite series of spheroïdal functions with known coefficients.

I-Problem Formulation

- We consider a monochromatic plane wave incident on a perfectly conducting obstacle of some form. The obstacle present a translational symmetry invariance along the z-axis.
- The wave field Ψ obeys to the two dimensional Helmholtz equation:

$$(\Delta + k^2)\Psi(x,y) = 0, \quad \Psi = \Psi^{\text{inc}} + \Psi^{\text{sc}}$$

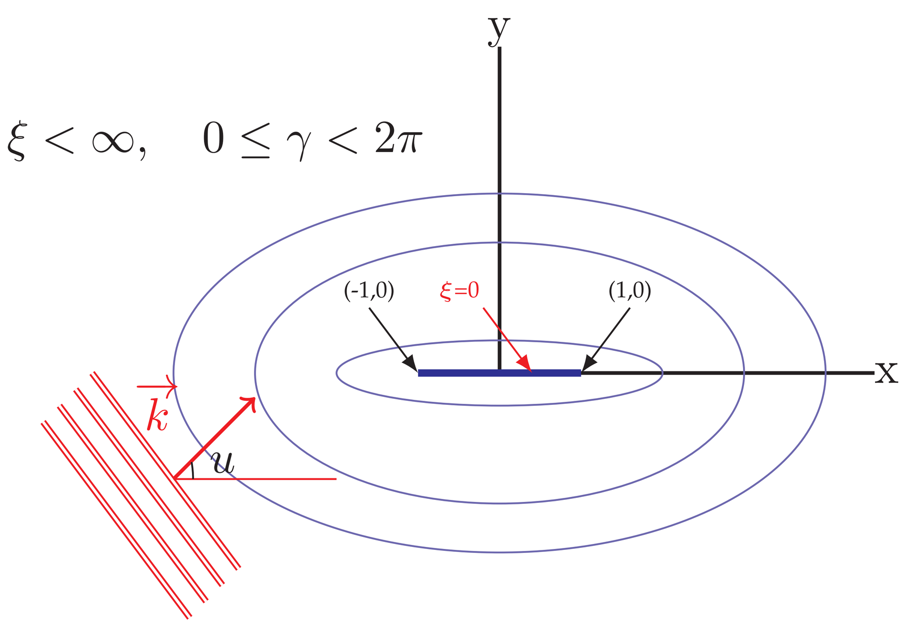
- Dirichlet boundary conditions on the conductor's surface ∂S:

$$\Psi|_{\partial S} = 0$$

- Sommerfeld radiation condition: the scattered wave is only outgoing at infinity.

II-Separation of variable in elliptic coordinate

$$0 \le \xi < \infty, \quad 0 \le \gamma < 2\pi$$



- Elliptic coordinates :

$$x = \cosh\xi\cos\gamma, \quad y = \sinh\xi\sin\gamma;$$

- Separation of variables: $(\Delta + k^2)X(\xi)Y(\gamma) = 0$

$$Y'' + (h - \tfrac{1}{2}k^2\cos 2\gamma)Y = 0 \qquad X'' - (h - \tfrac{1}{2}k^2\cosh 2\xi)X = 0$$

- Y obeys angular **Mathieu** equation [0]: Schrödinger equation in a periodic potential $\to$ Bloch's theorem ensures the existence of periodic solutions.
- By analytic continuation we have the radial functions $X(\xi) \equiv Y(i\xi)$.
- $Y(\gamma)$ = analogue of $\cos m\gamma$ and $\sin m\gamma$: $Ce_m(\gamma)$ and $Se_m(\gamma)$ m must be identified with the band index.
- $X(\xi)$ = analogue of $J_m(kr)$ and $Y_m(kr)$ with $r = e^{\xi}/2$: $Je_m^{(c,s)}(\xi)$ and $Ye_m^{(c,s)}(\xi)$.

III-Diffraction by a strip: exact solution

- By changing the parameter of the ellipse we can obtain the scaterred field by a ribbon [0]. The general solution of the Helmholtz equation must be written as a certain combination of Mathieu functions which respects the Sommerfeld radiation condition

$$\Psi(\xi,\gamma) = e^{ik(x\cos u + y\sin u)} + \sqrt{8\pi}\sum_m i^m \frac{Ce_m(\gamma)Ce_m(u)}{N_m} R_m^{(D)}\left[Je_m^{(c)}(\xi) + iYe_m^{(c)}(\xi)\right]; \qquad \int_0^{2\pi}[Ce_m]^2 d\gamma = N_m$$

- If the boundary condition is that the wave function Ψ is the zero at the ribbon, one gets:

$$\Psi(\xi = 0, \gamma) = 0 \quad \Longrightarrow \quad R_m^{(D)} = -\frac{Je_m^{(c)}(0)}{Je_m^{(c)}(0) + iYe_m^{(c)}(0)}$$

IV-Integral equation

- The propagator of the Helmholtz operator in two dimension is the Hankel function of first kind $H_0^{(1)}(k|x|)$. Thus, the Dirichlet boundary condition reads:

$$\int_{-1}^{1}\mu(t,u;k)H_0^{(1)}(k|x-t|)dt = -e^{ikx\cos u} = f(\cos u)$$

- Mathieu function solution is based on eigenfunction expansion

$$\mu_m\int_0^{\pi}H_0^{(1)}(k|\cos v - \cos w|)Ce_m(w)dw = Ce_m(v)$$

with known eigenvalue μ_m. If $f(\cos v) = \sum_m A_m Ce_m(v)$ then $\mu(\cos v) = \sum_m \mu_m A_m Ce_m(v)$
- μ is related to the normal derivative of the scaterred field. in elliptic coordinate this reads:

$$\mu(\gamma, u) = \frac{1}{2i\sin(\gamma)}\frac{\partial\Psi^{\text{sc}}(\xi,\gamma)}{\partial\xi}\bigg|_{\xi=0}$$

V-General integral equation and Latta's method

- We consider the integral equation for unknown function $g(t)$:

$$\int_{-1}^{1}K(|x-t|)g(t)dt = f(x),$$

with $K(w) = |w|^{\nu}K_{\nu}(\theta|w|)$ where $K_{\nu}(z)$ is the modified Bessel function of third kind.
- Diffraction on a strip corresponds to $\nu = 0$ and $\theta = -ik$.
- Latta's method [0] transforms an integral equation into a differential equation.
- It works only for kernel satisfying ODE with linear coefficients:

$$u\mathrm{L}K(u) + \mathrm{M}K(u) = 0, \quad L \text{ and } M \text{ linear differential operators.}$$

- Let Γ be the integral operator $\Gamma\star \equiv \int_{-1}^{1}K(|x-t|)\star dt$, the two important formulae of the method are the following:

$$\mathrm{L}\Gamma x f = x\mathrm{L}\Gamma f + \mathrm{M}\Gamma f,$$
$$\Gamma' y = \Gamma y' \text{ if } y(\pm 1) = 0.$$

VI-Scaling and Matrix equations

- Define $(\Gamma g_c)(x) = \cosh(\theta x)$ and $(\Gamma g_s)(x) = \sinh(\theta x)$. By use of Latta's equation one gets the first important differential equations coupling the two unknown:

$$\frac{\partial}{\partial t}\begin{pmatrix} g_c \\ g_s \end{pmatrix} = \mathbf{M}\begin{pmatrix} g_c \\ g_s \end{pmatrix}$$

where $\mathbf{M}$ is a matrix depending on t, ν, θ, and η which is defined as

$$\eta(\theta) \equiv \lim_{t\to 1}\frac{g_s(t,\theta)}{g_c(t,\theta)}$$

- Functions g_c and g_s depend on t and θ, $g_{c,s} = g_{c,s}(t,\theta)$
- General system of equations

$$\int_{L_1}^{L_2}K(|x-t|)G_{c,s}(t)dt = \begin{cases}\cosh k(x - \frac{1}{2}(L_1+L_2)) \\ \sinh k(x - \frac{1}{2}(L_1+L_2))\end{cases}$$

- $G_{c,s}$ are related to $g_{c,s}$ by a scaling transformation.
- Differentiation over strip ends and the use of Latta's equation leads to a second system of equation [0],[0]:

$$\frac{\partial}{\partial\theta}\begin{pmatrix} g_c \\ g_s \end{pmatrix} = \mathrm{N}\begin{pmatrix} g_c \\ g_s \end{pmatrix}$$

with N a matrix depending on t, θ and η.
- Compatibility condition

$$\frac{\partial}{\partial t}N - \frac{\partial}{\partial\theta}M = MN - NM$$

VII-Painlevé equation. Solution and Conclusion

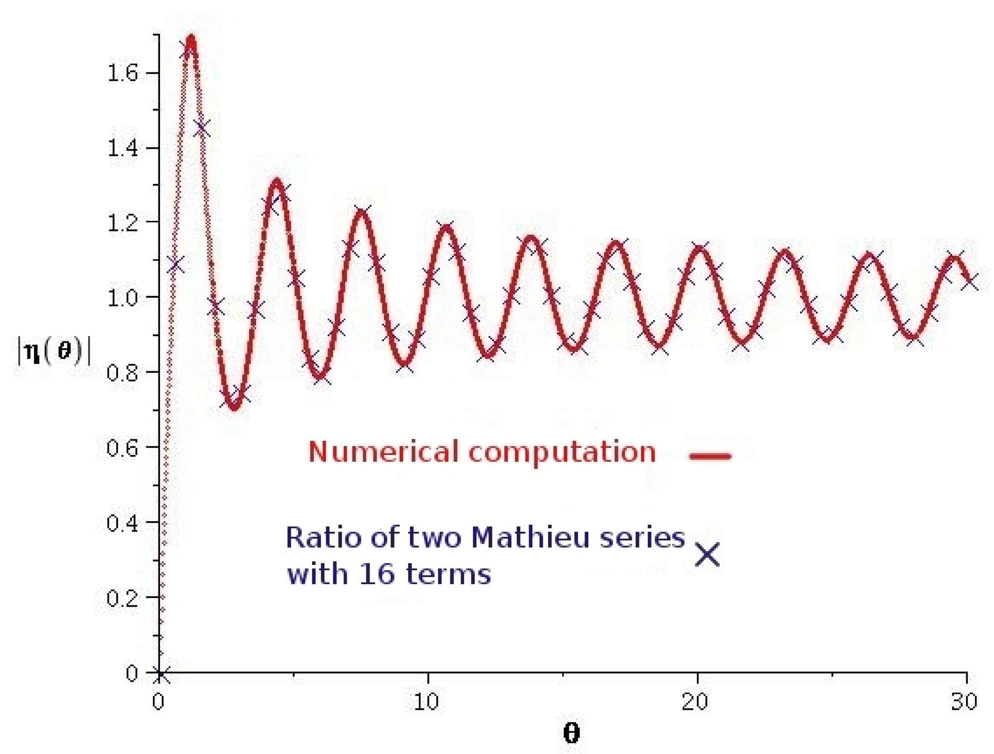
- Direct calculations proves that η obeys to the third Painlevé equation

$$\frac{d^2\eta}{d\theta^2} = \eta^{-1}\left(\frac{d\eta}{d\theta}\right)^2 - \theta^{-1}\frac{d\eta}{d\theta} - \frac{2\nu(1-\eta^2)}{\theta} + \eta^3 - \frac{1}{\eta}$$

- The small and large -θ behaviour are obtained from the behaviour of the integral kernel $K(w)$.
- For the diffraction problem $\nu = 0$, $\theta \to -i\theta$ and $\eta \to -i\eta$ we have

$$\eta(\theta) \underset{\theta\to 0}{\longrightarrow} -\theta\left(\ln\left(\frac{\theta}{4}\right) + \gamma\right) \qquad \gamma \equiv \text{Euler Mascheroni constant.}$$

$$\eta(\theta) = \lim_{t\to 1}\frac{1}{i}\frac{\mu(t,0;\theta) - \mu(t,\pi;\theta)}{\mu(t,0;\theta) + \mu(t,\pi;\theta)} = \frac{\sum_{m=0}^{\infty}\Gamma_{2m+1}(q)\,\mathrm{Ce}_{2m+1}^2(0,q)}{\sum_{m=0}^{\infty}\Gamma_{2m}(q)\,\mathrm{Ce}_{2m}^2(0,q)}, \ \theta = 2\sqrt{q}, \text{ coefficients } \Gamma_m \text{ known.}$$



- Special solutions of certain integral equations can be obtained from ODE.
- We generalize the result to the case $\nu \neq 0$. Mathieu functions are replaced by spheroïdal functions.
- Our results incorporate Myer's results [0].
- The positive–definiteness of the kernel forces this Painlevé III solution to be bounded for all positive arguments thus giving another proof of connection formulas for the Painlevé III equation in special cases.

References

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